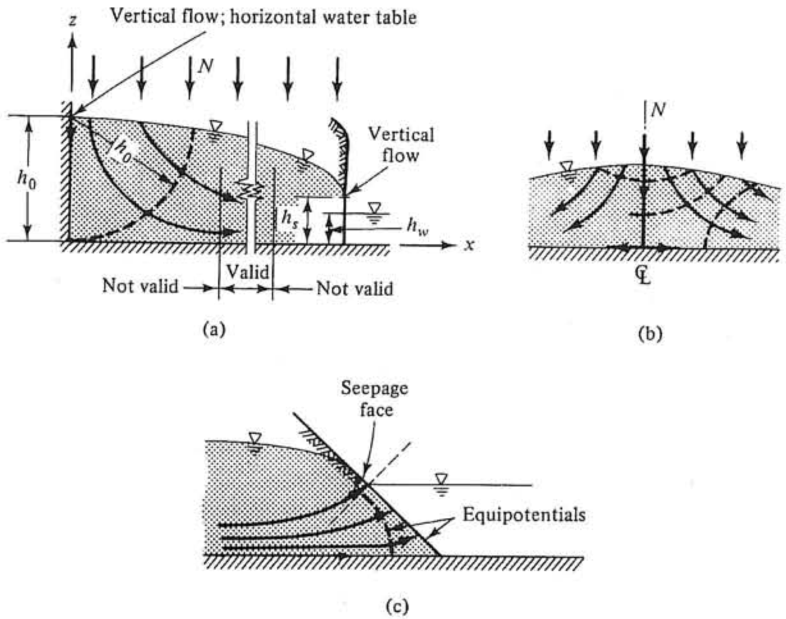

Figure 4-11 Regions where Dupuit assumptions are not valid.

neglected, before applying the Dupuit assumptions. Another case to which the Dupuit assumptions should be applied with care is that of unsteady flow in a decaying phreatic surface mound. Although no accretion takes place, yet at, and in the vicinity of, the crest the flow is vertically downward. At a distance of say 1.5 – 2 times the thickness of the flow, the approximation of vertical equipotentials is again valid.

In spite of what was said above, in regional studies, the Dupuit assumptions, because of their simplicity and the relatively small error involved, are usually applied also to those (relatively small) parts of an investigated region where they are not strictly applicable. One should, however, be careful in making use of results (say, water levels) derived for these parts of an investigated region.

CHAPTER

FIVE

MATHEMATICAL STATEMENT OF THE GROUNDWATER FORECASTING PROBLEM

The basic laws governing the flow of water in confined and phreatic aquifers are presented in the previous section. However, if we observe (4-9) closely, we notice that we actually have here one equation with two dependent variables: $\mathbf{q}(x, y, z, t)$ and $\phi(x, y, z, t)$, or three equations in four unknowns ϕ, q_x, q_y, q_z, if we refer to components of $\mathbf{q}$. This means that one additional equation is required in order to obtain a complete description of the flow regime in an aquifer. Similarly, we have $\mathbf{Q}'(x, y, t)$ and $\phi(x, y, t)$ in the single equation (4-27) and $\mathbf{Q}'(x, y, t)$ and $h(x, y, t)$ in the single equation (4-57). The additional basic law that we have to invoke is that of *conservation of matter, or mass*, which here takes the form of a *continuity equation*.

One should not be surprised that in Sec. 4-5 we did succeed in solving (4-57) for some simple cases. We actually used there the equation of continuity, which in that case took the simple form of $Q' = \text{const}$.

Our objective in what follows is to develop the continuity equations for different types of aquifers. The distribution of $\phi = \phi(\mathbf{x}, t)$ in an aquifer is obtained by solving these equations, subject to appropriate boundary and initial conditions.

We shall first consider the basic equations and boundary conditions for three-dimensional flows. Then, the equations for flow in aquifers will be developed for confined, leaky, and phreatic aquifers. We shall derive these (integrated, or averaged) aquifer equations in two ways. First, by merely assuming that the flow in an aquifer is essentially horizontal (see Secs 4-3 and 4-5), and writing a balance for a control volume which has the height of the saturated flow domain in the aquifer, and secondly, by integrating the point continuity equation over the vertical height of the aquifer. In this way, the conditions on the confined, leaky, or phreatic, upper and lower boundaries of the aquifers will be incorporated in the resulting integrated equations. It will be of interest to note that although the latter method is more rigorous, the results are identical.

With the material presented in this section, one should be able to state mathematically *any* groundwater flow problem. The problem of the movement of pollutants dissolved in the water is discussed in Chap. 7.

The determination of the future distribution of piezometric heads $\phi = \phi(\mathbf{x}, t)$, is, in fact, the solution to the groundwater forecasting problem referred to in Chap. 1. We are looking for future piezometric heads, or water levels, produced in a given (by geometry and properties) aquifer by any planned schedule of future pumping and artificial recharge activities and anticipated natural replenishment.

5-1 AQUIFER STORATIVITY

Specific Storativity

Let us start by introducing the concept of *effective stress* (or *intergranular stress*), first introduced by Terzaghi (1925).

Figure 5-1a shows a cross section through a confined aquifer. To simplify the discussion, we shall consider a granular non-cohesive matrix with grain sizes such that molecular and interparticle forces are negligible. Figure 5-1b shows the details at any internal horizontal elemental plane AB in an aquifer, whether confined or phreatic; it is also valid for an elemental surface of the impervious ceiling of an aquifer ($A'B'$).

The total load of soil and water (and actually also everything that adds load at the ground surface, including atmospheric pressure) above the considered plane is balanced by a stress (force per unit area) σ' in the solid matrix and by a pressure p in the water (Fig. 5-1b)

$$\sigma = \sigma' + p \tag{5-1}$$

where σ is the *total stress* resulting from the overburden load. Each of the three terms appearing in (5-1) is a force divided by the total area, A, of the considered plane. Strictly speaking, we should have taken into account the fact that whereas

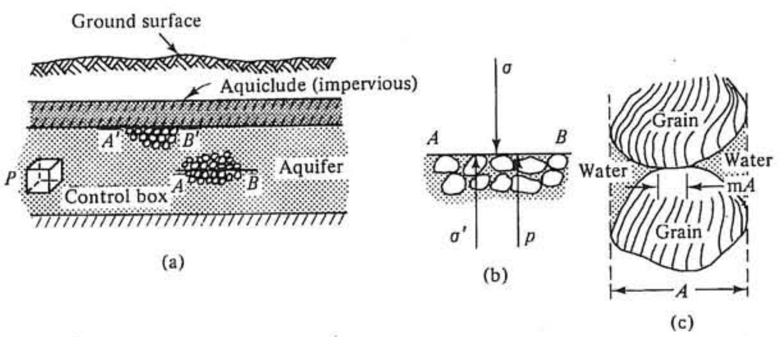

Figure 5-1 Pressure and intergranular stress in an aquifer.

σ acts over the entire area under consideration, the water pressure acts only over part of A and so does the force carried by the solid matrix. Nevertheless, it can be shown (e.g., Lambe and Whitman, 1969; Bear, 1972, p. 54) that the *effective stress* σ', as defined above, is a good approximation of the stress transmitted through the skeleton (and hence it is also called *intergranular stress*), and that we may assume that indeed p acts over the entire area A. In (5-1), positive σ and σ' denote compression.

Equation (5-1) is derived by considering vertical forces only. However, the discussion can be extended to the general case of three-dimensional space (e.g., Bear, 1972, p. 55; Verruijt, 1965, 1969).

When changes in the overburden load take place, changes will be produced also in σ' and p

$$d\sigma = d\sigma' + dp \tag{5-2}$$

If we keep σ = const., but change the pressure, for example by pumping from the aquifer, or by artificially recharging it, we have

$$d\sigma = 0 = d\sigma' + dp, \qquad d\sigma' = -dp \tag{5-3}$$

which means that a corresponding change is produced in the intergranular stress. Thus, *a reduction of water pressure by pumping from a well results in an increase in the load borne by the solid skeleton of the aquifer.*

Now, the water in the aquifer is compressible. Although this compressibility is small, it plays an important role mainly in confined aquifers. We define a coefficient of compressibility of water, β, by

$$\beta = -\frac{1}{U_w}\frac{\partial U_w}{\partial p} = \frac{1}{\rho}\frac{\partial \rho}{\partial p} \tag{5-4}$$

where U_w and ρ are volume and density, respectively, of a given mass of water subjected to pressure changes. The minus sign indicates a decrease in volume as pressure increases. For β independent of pressure, we obtain from (5-4)

$$U_w = U_{w0}\exp[-\beta(p - p_0)]; \qquad \rho = \rho_0 \exp[\beta(p - p_0)] \tag{5-5}$$

where U_{w0} and ρ_0 correspond to the reference pressure p_0.

The solid matrix of the aquifer is elastic and not rigid. By subjecting it to a change in the intergranular stress, it will undergo deformation. This deformation involves a movement of the solid, or the solid particles and their rearrangement, such that the porosity of the porous medium is changed. We assume that the elasticity of the solid or the solid particles is much smaller (relative to the solid matrix as a whole), so that their volume remains unchanged.

The elastic property of the solid matrix is expressed by its coefficient of compressibility, α, defined by

$$\alpha = -\frac{1}{U_b}\frac{\partial U_b}{\partial \sigma'}, \tag{5-6}$$

where U_b is the bulk volume of a porous medium. As emphasized above, we are considering here only vertical compressibility, the lateral deformation in the aquifer is assumed negligible.

Since the volume of solids U_s in U_b remains constant, we have

$$U_s \equiv (1-n)\,U_b = \text{const.}; \qquad \frac{\partial U_s}{\partial \sigma'} = 0; \qquad \frac{1}{U_b}\frac{\partial U_b}{\partial \sigma'} = \frac{1}{(1-n)}\frac{\partial n}{\partial \sigma'};$$

$$\alpha = -\frac{1}{1-n}\frac{\partial n}{\partial \sigma'} = \frac{1}{1-n}\frac{\partial n}{\partial p} \qquad (5\text{-}7)$$

which relates α to the changes in porosity, n, resulting from changes in water pressure.

Consider now the vicinity of a point in an aquifer, where water pressure is reduced by pumping. As indicated by (5-3), this results in an increase in the intergranular stress transmitted by the solid skeleton of the aquifer. This, in turn, causes the aquifer to be compacted, reducing its porosity. At the same time, as a result of pressure reduction, the water will expand according to (5-4). Together, the two effects—the slight expansion of water and the small reduction in porosity—cause a certain amount of water to be released from storage in an aquifer. Thus, by releasing water from storage in an aquifer, we produce in it a reduction in water pressure. Conversely, in response to adding water to a unit volume of aquifer, the pressure in it will rise, accompanied by a reduction in the intergranular stress, which, in turn, increases the porosity. If we assume both water and solid matrix to be perfectly elastic, within the range of the considered changes, the two processes are reversible. In reality, however, changes in a granular matrix are irreversible (see Sec. 5-11).

Based on the above considerations, we can now define a *specific storativity*, S_{0p}, of the porous medium of an aquifer as the volume of water released from storage (or added to it) in a unit volume of aquifer per unit decline (or rise) in pressure

$$S_{0p} = \Delta U_w / U_b \Delta p \qquad (5\text{-}8)$$

or, per unit change in the piezometric head ϕ^* (defined by (4-4))

$$S_0 \equiv S_{0\phi^*} = \Delta U_w / U_b \Delta \phi^* \qquad (5\text{-}9)$$

S_0 has the dimensions of L^{-1}. From (5-9) it follows that by adding a volume ΔU_w to a volume U_b of aquifer, the piezometric head there will rise by $\Delta\phi^* = \Delta U_w / U_b S_0$.

One should note that S_{0p} and $S_{0\phi^*}$ are actually *defined* by (5-8) and (5-9), respectively, without analyzing their internal relationship to the compressibilities of water and solid matrix.

Aquifer Storativity

In a similar way we can define a *storativity for a confined aquifer*, S, as the volume of water released from storage (or added to it) per unit horizontal area of aquifer and per unit decline (or rise) of piezometric head, ϕ

$$S = \Delta U_w / A \Delta \phi \qquad (5\text{-}10)$$

S is dimensionless. The reasons for the relationship between the amount of water released and the change of head are now obvious. The volume of aquifer from which water is released is $A \times B$, where A is horizontal area and B is the thickness of the confined aquifer (Fig. 5-2).

It is important to understand that like T, S is an aquifer property. If we work under the assumption of essentially horizontal flow in an aquifer (Sec. 2-4), we should use the parameters T and S. If, however, we wish to consider three-dimensional flow in an aquifer, we should use the parameters K and S_0. Although it is possible to relate K to T by $T = KB$ and S_0 to S by $S = S_0 B$, one should avoid mixing the two concepts, as in principle they are related to two different flow models.

We can also define a storage coefficient for a phreatic aquifer. Consider a unit (horizontal) area of a phreatic aquifer. The volume of water stored in a phreatic aquifer is indicated by the water table (see Sec. 5-4). If, as a result of the flow in the aquifer, a volume of water will leave this area in excess of the volume of water entering it, the water table will drop. We may define the *storativity of a phreatic aquifer* in the same way as we defined above the storativity of a confined aquifer, except that here the drop, Δh, is of the water table (Fig. 5-3)

$$S = \Delta U_w / A \Delta h \qquad (5\text{-}11)$$

In spite of the similarity in the definition, the storativity in the two types of aquifer is due to different reasons. In a confined aquifer, it is the outcome of water and matrix compressibility. In a phreatic aquifer, water is mostly drained

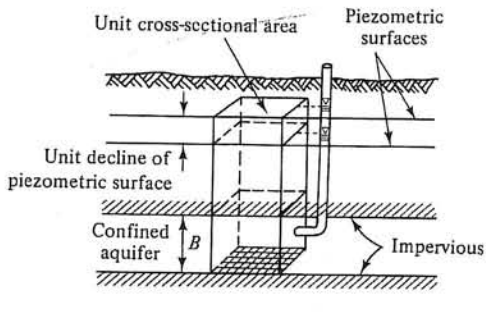

Figure 5-2 Definition sketch for storativity in a confined aquifer.

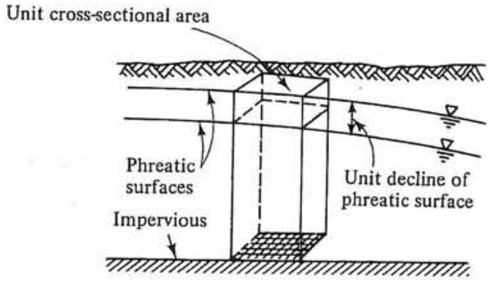

Figure 5-3 Definition sketch for storativity in a phreatic aquifer.

from the volume of pore space between the two positions of the phreatic surface. The storativity of a phreatic aquifer is, therefore, sometimes referred to as *specific yield*, S_y; it gives the yield of an aquifer per unit area and unit drop of the water table (see further discussion in Sec. 6-1).

Recalling that actually the water table is an approximate concept, we understand that water is actually being drained from the entire column of soil up to the ground surface. Bear (1972, p. 485) shows that when the soil is homogeneous and the fluctuating water table is sufficiently deep, the above definition for specific yield still holds (see Sec. 6-1).

One should be careful not to identify the specific yield with the porosity of a phreatic aquifer. As water is being drained from the interstices of the soil, the drainage is never a complete one. A certain amount of water is retained in the soil against gravity by capillary forces. After drainage has stopped, the volume of water retained in an aquifer per unit (horizontal) area and unit drop of the water table is called *specific retention*, S_r. Thus

$$S_y + S_r = n \tag{5-12}$$

For this reason S_y ($<n$) is sometimes called *effective porosity*. Here, again, one should note that we have been referring to the approximate concept of a water table. However, for a homogeneous soil and a sufficiently deep water table, the above definition for S_r holds (see Sec. 6-1).

Figure 5-4 shows the relationships between S_y, S_r, and particle size.

When drainage occurs, it takes time for the water to flow, partly under unsaturated flow conditions, out of the soil volume between two positions of a water table, at t and at $t + \Delta t$. This is especially true if the lowering of the water table is rapid. Under such conditions, the specific yield becomes time dependent, gradually approaching its ultimate value (Fig. 5-5). When the water level is rising or falling slowly, the changes in moisture distribution have time to adjust continuously and the time lag vanishes. This phenomenon of time dependency of the

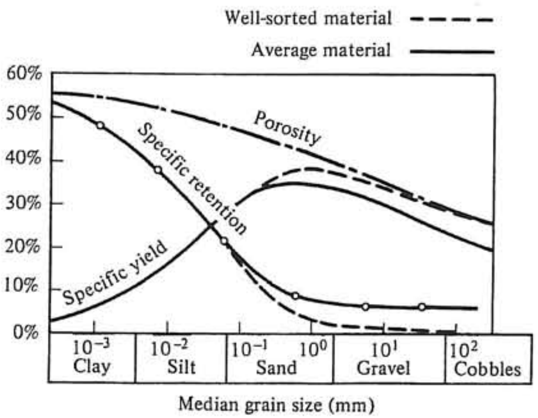

Figure 5-4 Relationship between specific yield and grain size *(from Conkling et. al., 1934, as modified by Davis and DeWiest, 1966)*.

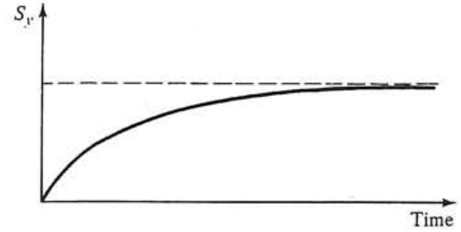

Figure 5-5 Time-dependent specific yield.

specific yield should not be overlooked in the analysis of pumping tests. water balances for short periods, etc. (see Neuman, 1972).

When the water table is lowered, the pressure drops throughout the aquifer below it. In principle, this pressure drop should cause water to be released from storage in the aquifer, due to the elastic properties of the aquifer and the water. However, when we calculate the total volume of water released from storage in the aquifer per unit area and unit decline of head: $(\Delta U_w)_1 = S_0 h$ due to the elastic storage and $(\Delta U_w)_2 = S_y$ due to the actual drainage of water from the pore space, we have $S_0 h \ll S_y$ so that $(\Delta U_w)_1$ can be neglected (see further discussion on this point in Sec. 5-2).

Typical values of S in a confined aquifer are of the order of 10^{-4}–10^{-6}, roughly 40 percent of which result from the expansion of the water and 60 percent from the compression of the medium. In a sandy phreatic aquifer, we may have S_0 of the order 10^{-7} cm^{-1}, whereas S_y may be 20–30 percent (see Fig. 5-4).

We shall return to the definition of aquifer storativity, both for a confined aquifer and for a phreatic one, in Sec. 5-2 where the aquifer equations will be derived by averaging the three-dimensional flow equations along the vertical.

5-2 BASIC CONTINUITY EQUATION

Mass Conservation Equation

In this section, we shall develop the basic equation describing three-dimension flow in a porous medium. One way of deriving the basic mass balance equation is given in App. A-6, leading to (A-24)

$$\frac{\partial(n\rho)}{\partial t} = -\operatorname{div} \rho \mathbf{q} \tag{5-13}$$

Nevertheless, let us derive this equation by using more elementary considerations.

Consider a *control volume* (*or control box*) having the shape of a rectangular parallel-piped box of dimensions δx, δy, δz centered at some point $P(x, y, z)$ inside the flow domain in an aquifer (Figs. 5-1 and 5-6). A control box may have any arbitrary shape, but once its shape and position in space have been fixed, they remain unchanged during the flow, although the amount and identity of the material in it may change with time. In the present analysis, water and solids