Tentative program

8.15 – 8.30	Group work – Discuss questions and prepare a talk of definitions
	Question :10 to 18
8.30 – 8.45	Group wise presentation
8.45 – 10.00	Numerical modelling and formulation of finite difference equations
10.00 – 10.30	Break
10.30 – 11.00	Numerical modelling and formulation of finite difference equations
11.00 – 13.00	Exercises 2
13.00 – 14.00	Implementation of boundary conditions
14.00 – 16.00	Exercise 2

Questions

1. How is the saturated zone defined
2. What is an aquifer?
3. What is an aquitard?
4. What is an unconfined aquifer?
5. What is a confined aquifer?
6. What is a conceptual model?
7. How is specific storage defined ?
8. How is specific yield defined?
9. How is storage coefficient defined?
10. What is hydraulic conductivity?
11. What is the difference between hydraulic conductivity and permeability ?
12. What is transmissivity?
13. What is the typical range for hydraulic conductivity for sand?
14. What is the typical range for hydraulic conductivity for clay?
15. How is hydraulic head defined?
16. What is the Darcy velocity?
17. What is the pore water velocity?
18. Which two equations is joined in the governing equation for groundwater flow?

Mathematical basis for groundwater flow

The equation of continuity equation combined with the Darcy equation

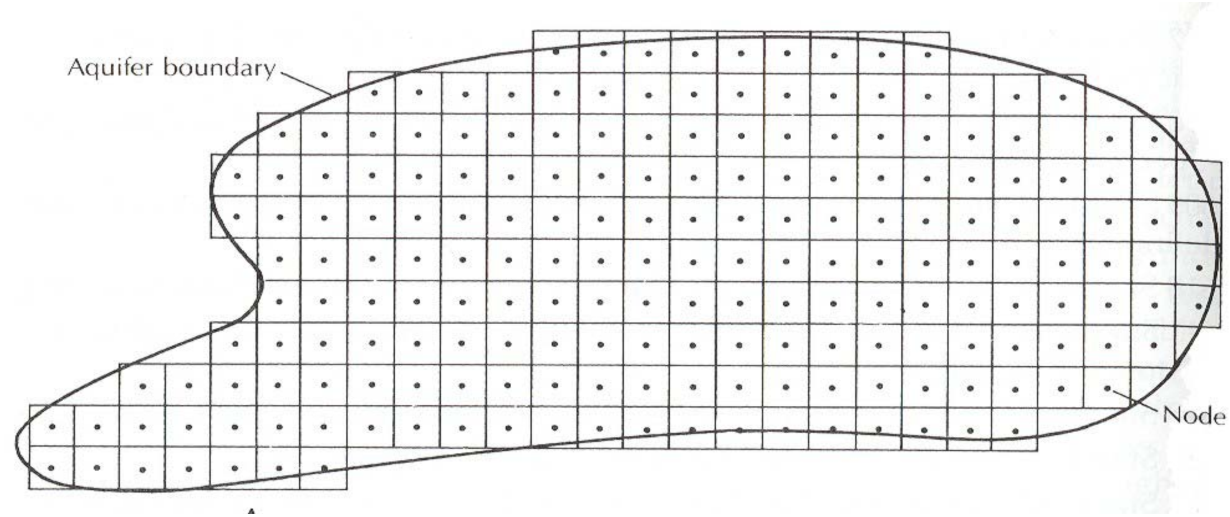
$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} - R = S_S \frac{\partial h}{\partial t}$$

K_x, K_y, K_z	- hydraulic conductivity
h	- *total head*
R	- *Sink/source term*
S_S	- *Specific storage*

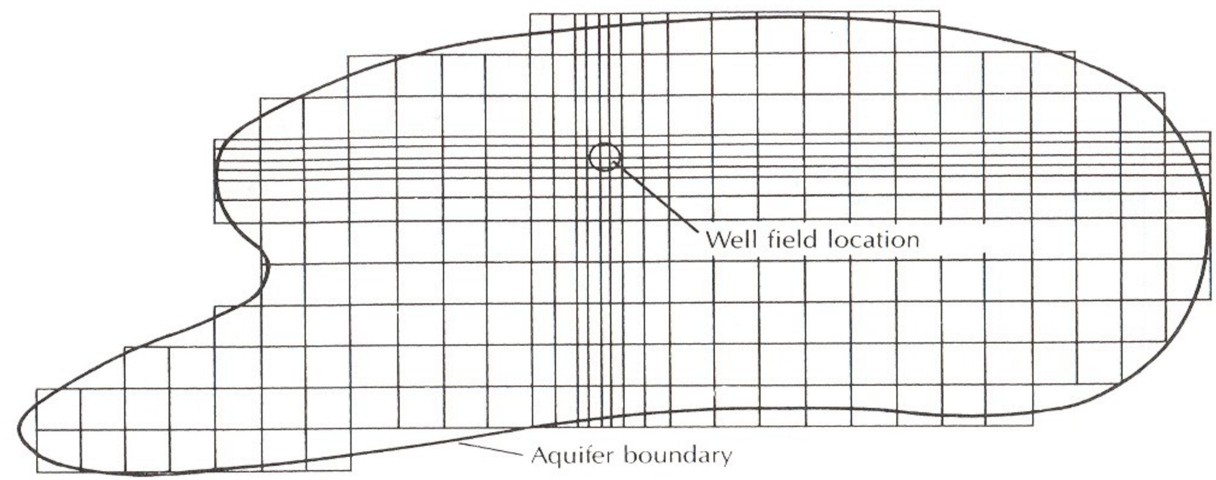
Numerical models

Finite difference modeller:

square cells

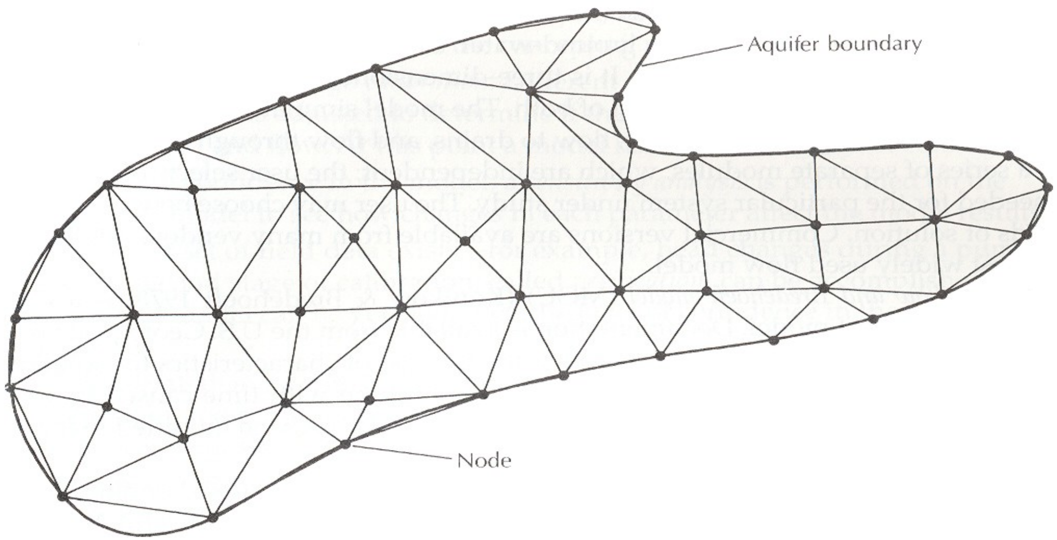


rectangular cells



Numerical models

Finite element models:



Numerical models

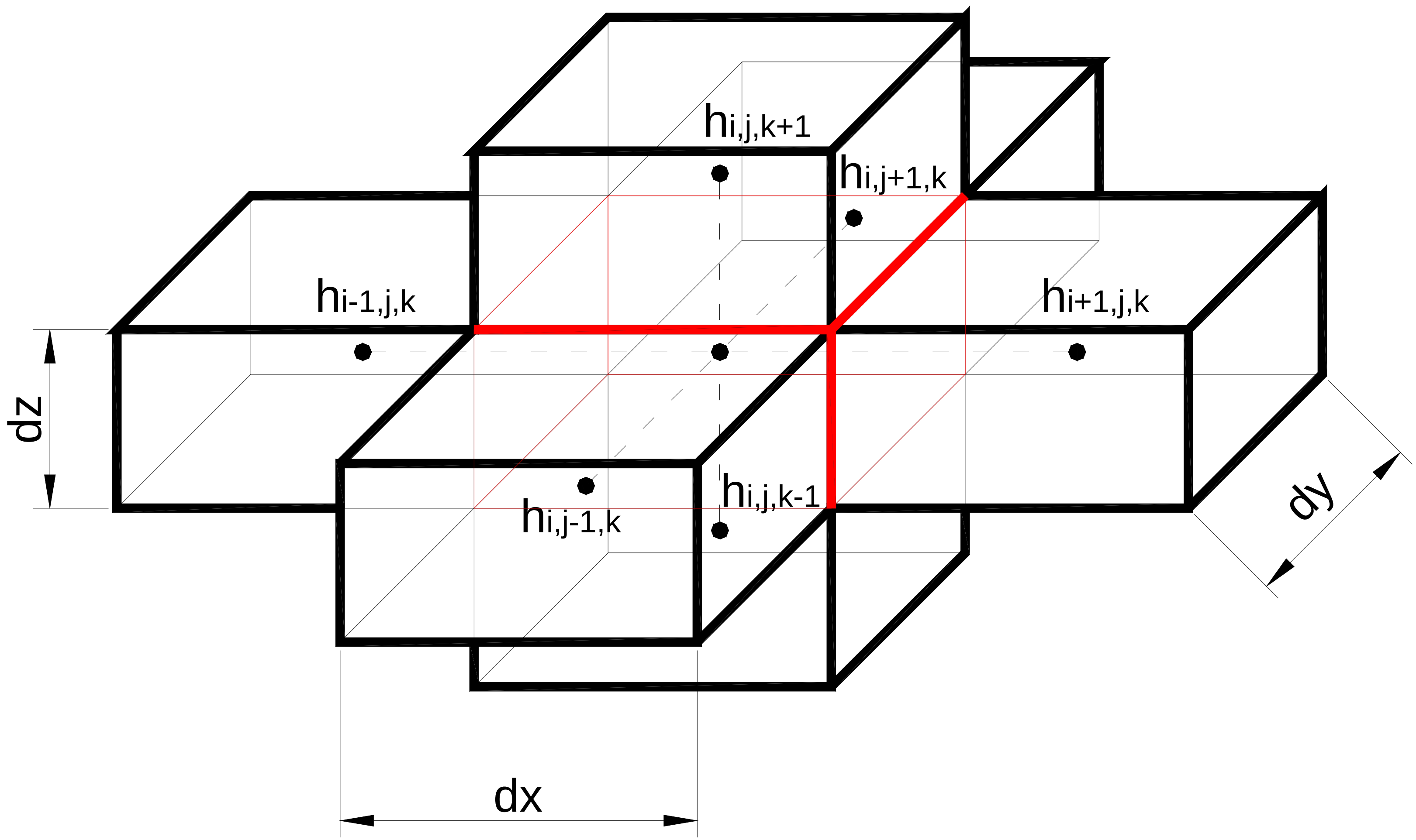
Spatial discretisation - consider:

- Purpose of modelling
- Variations in hydraulic head (Near well fields, rivers, faults, etc.)
- Geological layers
- Locations of well fields
- Exchange with rivers
- Computational resources
- …

Discretisation in time - consider:

- Purpose of modelling
- Head/flow variation at boundaries
- Timescale of involved processes
- ….

Description of Finite Difference equations for 3D groundwater flow



Description of Finite Difference equations for 3D groundwater flow

Equation of continuity:

$$\text{in - out} = \Delta sto$$

$$\Delta t\left(Q_x^- - Q_x^+ + Q_y^- - Q_y^+ + Q_z^- - Q_z^+ - R\right) = \Delta sto$$

The Darcy equation:

$$Q_x^- = -\frac{h_{i,j,k}^{n+1} - h_{i-1,j,k}^{n+1}}{\Delta x} Kx_{i-1/2,j,k} \Delta y \Delta z \qquad Q_x^+ = -\frac{h_{i+1,j,k}^{n+1} - h_{i,j,k}^{n+1}}{\Delta x} Kx_{i+1/2,j,k} \Delta y \Delta z$$

$$Q_y^- = -\frac{h_{i,j,k}^{n+1} - h_{i,j-1,k}^{n+1}}{\Delta y} Ky_{i,j-1/2,k} \Delta x \Delta z \qquad Q_y^+ = -\frac{h_{i,j+1,k}^{n+1} - h_{i,j,k}^{n+1}}{\Delta y} Ky_{i,j+1/2,k} \Delta x \Delta z$$

$$Q_z^- = -\frac{h_{i,j,k}^{n+1} - h_{i,j,k-1}^{n+1}}{\Delta z} Kz_{i,j,k-1/2} \Delta x \Delta y \qquad Q_z^+ = -\frac{h_{i,j,k+1}^{n+1} - h_{i,j,k}^{n+1}}{\Delta z} Kz_{i,j,k+1/2} \Delta x \Delta y$$

Description of Finite Difference equations for 3D groundwater flow

Effective conductivity on the cell edges :

$$Kx_{i-\frac{1}{2},j,k} = \frac{2}{\dfrac{1}{Kx_{i-1,j,k}} + \dfrac{1}{Kx_{i,j,k}}}, \quad Kx_{i+\frac{1}{2},j,k} = \frac{2}{\dfrac{1}{Kx_{i+1,j,k}} + \dfrac{1}{Kx_{i,j,k}}}$$

$$Ky_{i,j-\frac{1}{2},k} = \frac{2}{\dfrac{1}{Ky_{i,j-1,k}} + \dfrac{1}{Ky_{i,j,k}}}, \quad Ky_{i,j+\frac{1}{2},k} = \frac{2}{\dfrac{1}{Ky_{i,j+1,k}} + \dfrac{1}{Ky_{i,j,k}}}$$

$$Kz_{i,j,k-\frac{1}{2}} = \frac{dz_{i,j,k-1} + dz_{i,j,k}}{\dfrac{dz_{i,j,k-1}}{Kz_{i,j,k-1}} + \dfrac{dz_{i,j,k}}{Kz_{i,j,k}}}, \quad Kz_{i,j,k+\frac{1}{2}} = \frac{dz_{i,j,k+1} + dz_{i,j,k}}{\dfrac{dz_{i,j,k+1}}{Kz_{i,j,k+1}} + \dfrac{dz_{i,j,k}}{Kz_{i,j,k}}}$$

Storage:

$$\Delta sto = \frac{h_{i,j,k}^{n+1} - h_{i,j,k}^{n}}{\Delta t} S \Delta x \Delta y = \frac{h_{i,j,k}^{n+1} - h_{i,j,k}^{n}}{\Delta t} S_s \Delta z \Delta x \Delta y$$

Description of Finite Difference equations for 3D groundwater flow

Equation of continuity and the Darcy equation

$$-\frac{h_{i,j,k}^{n+1}-h_{i-1,j,k}^{n+1}}{\Delta x}Kx_{i-1/2,j,k}\Delta y\Delta z+\frac{h_{i+1,j,k}^{n+1}-h_{i,j,k}^{n+1}}{\Delta x}Kx_{i+1/2,j,k}\Delta y\Delta z-\frac{h_{i,j,k}^{n+1}-h_{i,j-1,k}^{n+1}}{\Delta y}Ky_{i,j-1/2,k}\Delta x\Delta z+\frac{h_{i,j+1,k}^{n+1}-h_{i,j,k}^{n+1}}{\Delta y}Ky_{i,j+1/2,k}\Delta x\Delta z$$

$$-\frac{h_{i,j,k}^{n+1}-h_{i,j,k-1}^{n+1}}{\Delta z}Kz_{i,j,k-1/2}\Delta x\Delta y+\frac{h_{i,j,k+1}^{n+1}-h_{i,j,k}^{n+1}}{\Delta z}Kz_{i,j,k+1/2}\Delta x\Delta y-R=\frac{h_{i,j,k}^{n+1}-h_{i,j,k}^{n}}{\Delta t}S_S\Delta x\Delta y\Delta z$$

$$\Downarrow$$

$$\left(Kx_{i+1/2,j,k}\frac{\Delta y\Delta z}{\Delta x}\right)h_{i+1,j,k}^{n+1}+\left(Kx_{i-1/2,j,k}\frac{\Delta y\Delta z}{\Delta x}\right)h_{i-1,j,k}^{n+1}+\left(Ky_{i,j+1/2,k}\frac{\Delta x\Delta z}{\Delta y}\right)h_{i,j+1,k}^{n+1}+\left(Ky_{i,j-1/2,k}\frac{\Delta x\Delta z}{\Delta y}\right)h_{i,j-1,k}^{n+1}$$

$$+\left(Kz_{i,j,k+1/2}\frac{\Delta x\Delta y}{\Delta z}\right)h_{i,j,k+1}^{n+1}+\left(Kz_{i,j,k-1/2}\frac{\Delta x\Delta y}{\Delta z}\right)h_{i,j,k-1}^{n+1}$$

$$+\left(-Kx_{i-1/2,j,k}\frac{\Delta y\Delta z}{\Delta x}-Kx_{i+1/2,j,k}\frac{\Delta y\Delta z}{\Delta x}-Ky_{i,j-1/2,k}\frac{\Delta x\Delta z}{\Delta y}-Ky_{i,j+1/2,k}\frac{\Delta x\Delta z}{\Delta y}-Kz_{i,j,k-1/2}\frac{\Delta x\Delta y}{\Delta z}-Kz_{i,j,k+1/2}\frac{\Delta x\Delta y}{\Delta z}-\frac{S_S\Delta x\Delta y\Delta z}{\Delta t}\right)h_{i,j}^{n+}$$

$$=R-\frac{h_{i,j,k}^{n}}{\Delta t}S_S\Delta x\Delta y\Delta y$$

$$\Downarrow$$

$$Ah_{i+1,j,k}^{n+1}+Bh_{i-1,j,k}^{n+1}+Ch_{i,j+1,k}^{n+1}+Dh_{i,j-1,k}^{n+1}+Eh_{i,j,k+1}^{n+1}+Fh_{i,j,k-1}^{n+1}+Gh_{i,j,k}^{n+1}=H$$

Description of Finite Difference equations for 1D groundwater flow

Equation of continuity and the Darcy equation

$$-\frac{h_i^{n+1}-h_{i-1}^{n+1}}{\Delta x}Kx_{i-1/2}\Delta y\Delta z+\frac{h_{i+1}^{n+1}-h_i^{n+1}}{\Delta x}Kx_{i+1/2}\Delta y\Delta z-R=\frac{h_i^{n+1}-h_i^{n}}{\Delta t}S_S\Delta x\Delta y\Delta z$$

$$\Downarrow$$

$$-\left(h_i^{n+1}-h_{i-1}^{n+1}\right)Kx_{i-1/2}+\left(h_{i+1}^{n+1}-h_i^{n+1}\right)Kx_{i+1/2}-\frac{\Delta xR}{\Delta y\Delta z}=\frac{h_i^{n+1}-h_i^{n}}{\Delta t}S_S\Delta x\Delta x$$

$$\Downarrow$$

$$\left(Kx_{i+1/2}\right)h_{i+1}^{n+1}+\left(Kx_{i-1/2}\right)h_{i-1}^{n+1}+\left(-Kx_{i-1/2}-Kx_{i+1/2}-\frac{S_S\Delta x\Delta x}{\Delta t}\right)h_i^{n+1}$$

$$=R\frac{\Delta x}{\Delta y\Delta z}-\frac{h_i^{n}}{\Delta t}S\Delta x\Delta x$$

$$\Downarrow$$

$$A_ih_{i-1}^{n+1}+B_ih_i^{n+1}+C_ih_{i+1}^{n+1}=D_i$$

Description of Finite Difference equations for 1D groundwater flow

The *n* finite difference equations:

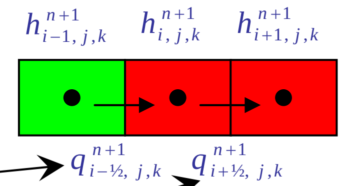
$$\begin{vmatrix} B_1 & C_1 & & \cdots & & \\ A_2 & B_2 & C_2 & & & \\ \vdots & & \ddots & & & \\ & & A_{n-1} & B_{n-1} & C_{n-1} \\ & & & A_n & B_n \end{vmatrix} \begin{vmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \\ h_n \end{vmatrix} = \begin{vmatrix} D_1 \\ D_2 \\ \vdots \\ D_{n-1} \\ D_n \end{vmatrix}$$

$\Downarrow$

$$h^{n+1}_{i-1,j,k} = \text{constant}$$

Implementation of boundary conditions

Prescribed head or Dirichlets condition
(surface water body, measured groundwater head)



1D FD equations

$$-\frac{h^{n+1}_{i,j,k} - h^{n+1}_{i-1,j,k}}{\Delta x} Kx_{i-\frac{1}{2},j,k}\Delta y\Delta z + \frac{h^{n+1}_{i+1,j,k}-h^{n+1}_{i,j,k}}{\Delta x} Kx_{i+\frac{1}{2},j,k}\Delta y \Delta z - R = \frac{h^{n+1}_{i,j}-h^{n}_{i,j}}{\Delta t} S_S \Delta x\Delta y\Delta z$$

$\Downarrow$ $\bullet\ \Delta x$ and $/(\Delta y \Delta z)$

$$-\left(h^{n+1}_{i,j,k} - h^{n+1}_{i-1,j,k}\right)Kx_{i-\frac{1}{2},j,k} + \left(h^{n+1}_{i+1,j,k}-h^{n+1}_{i,j,k}\right)Kx_{i+\frac{1}{2},j,k} - \frac{\Delta x R}{\Delta y\Delta z} = \frac{h^{n+1}_{i,j}-h^{n}_{i,j}}{\Delta t}S_S\Delta x\Delta x$$

$\Downarrow$

$$\left(Kx_{i+\frac{1}{2},j,k}\right)h^{n+1}_{i+1,j,k} + \left(-Kx_{i-\frac{1}{2},j,k} - Kx_{i+\frac{1}{2},j,k} - \frac{S_S\Delta x\Delta x}{\Delta t}\right)h^{n+1}_{i,j,k}$$

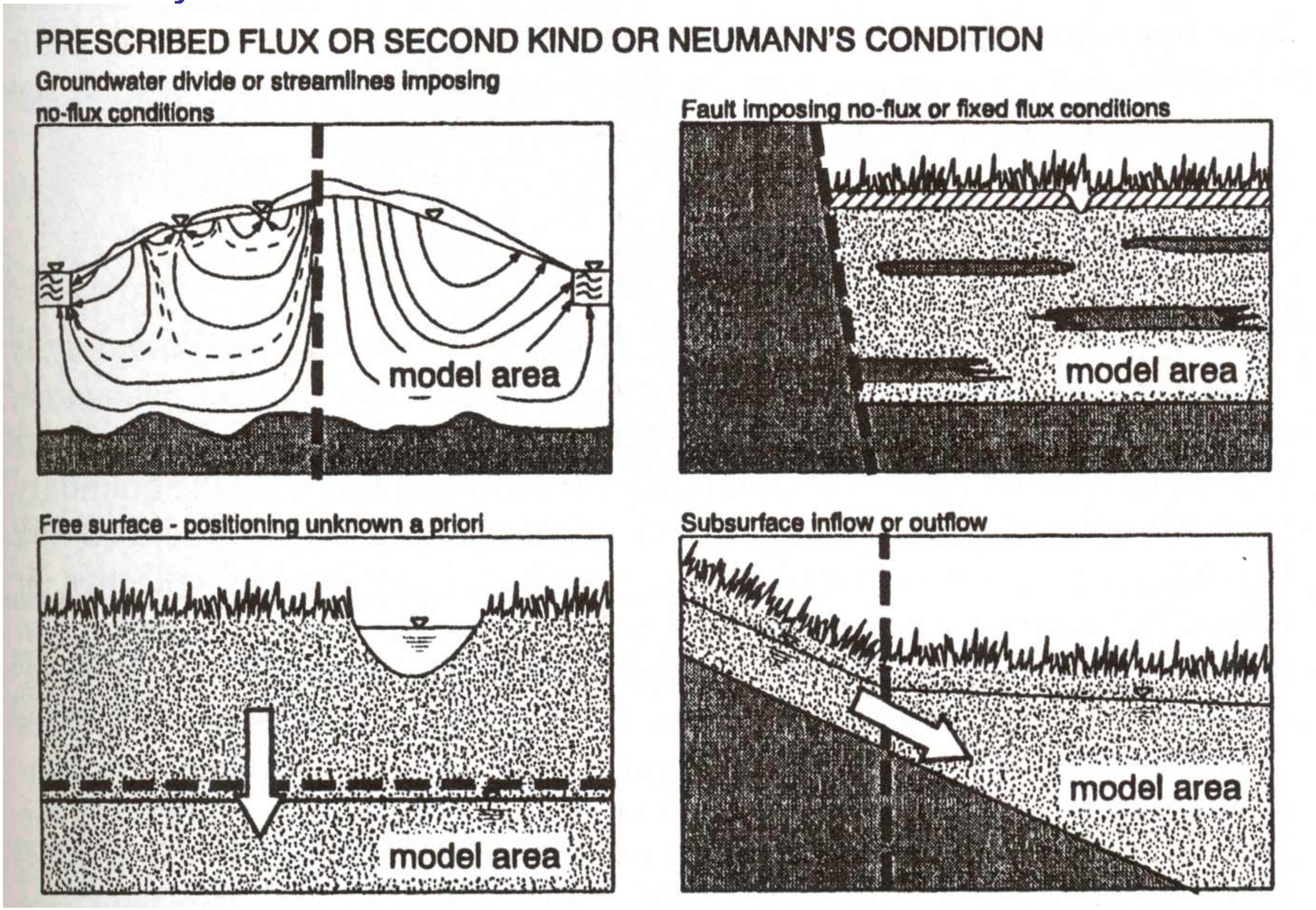
$$= R\frac{\Delta x}{\Delta y\Delta z} - \frac{h^{n}_{i,j,k}}{\Delta t}S\Delta x\Delta x - \left(Kx_{i-\frac{1}{2},j,k}\right)h^{n+1}_{i-1,j,k}$$

$\Downarrow$

$$Ah^{n+1}_{i+1,j,k} + Ch^{n+1}_{i,j,k} = D - Bh^{n+1}_{i-1,j,k}$$

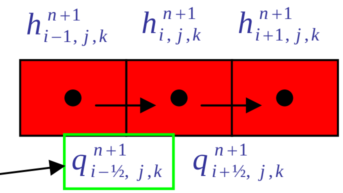
All known terms

Boundary conditions



Boundary conditions

Prescribed flux or Neumann's condition (Groundwater divide, subsurface inflow/outflow, etc.)



$$q_{i-\frac{1}{2},j,k} = \frac{h^{n+1}_{i,j,k} - h^{n+1}_{i-1,j,k}}{\Delta x} Kx_{i-\frac{1}{2},j,k} \Delta y \Delta z = \text{constant}$$

1D FD equations

$$q_{i-\frac{1}{2},j,k} + \frac{h^{n+1}_{i+1,j,k} - h^{n+1}_{i,j,k}}{\Delta x} Kx_{i+\frac{1}{2},j,k} \Delta y \Delta z - R = \frac{h^{n+1}_{i,j} - h^{n}_{i,j}}{\Delta t} S_S \Delta x \Delta y \Delta z$$

$$\Downarrow$$

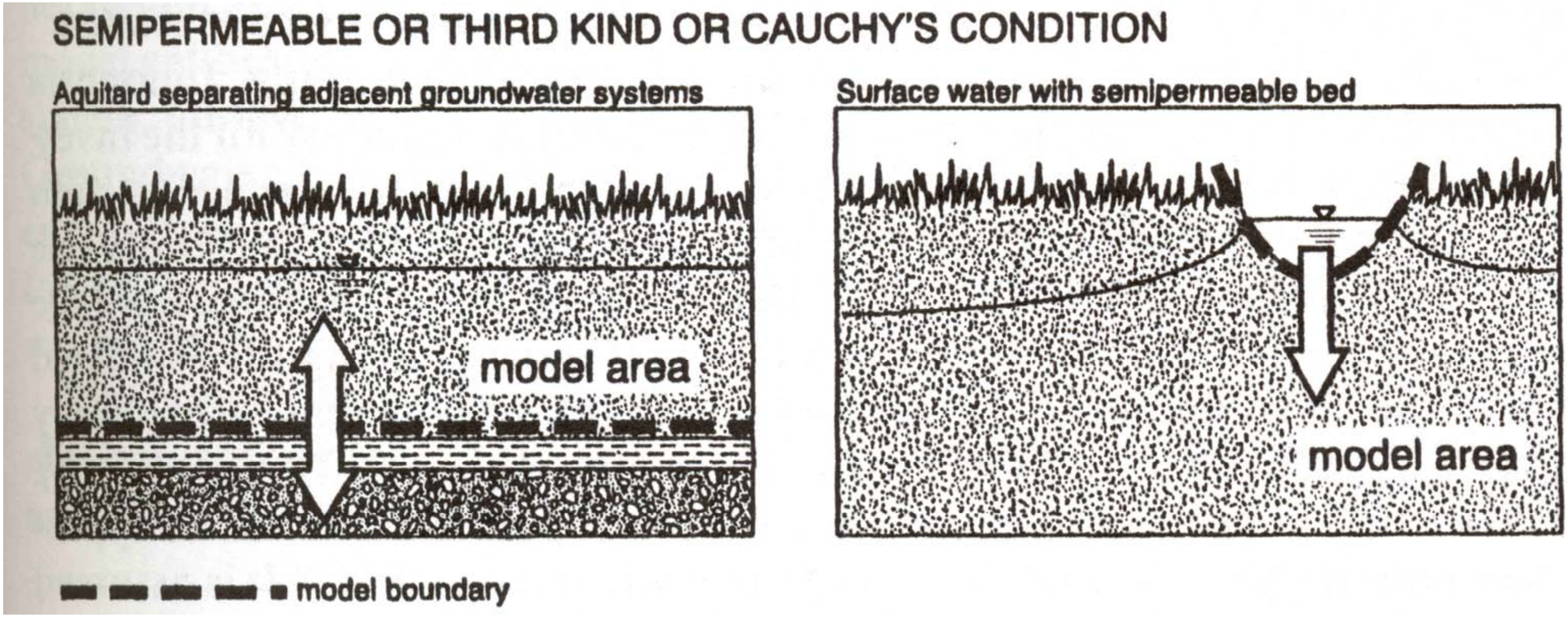
$$\left(Kx_{i+\frac{1}{2},j,k}\right) h^{n+1}_{i+1,j,k} - \left(Kx_{i+\frac{1}{2},j,k}\right) h^{n+1}_{i,j,k} = R \frac{\Delta x}{\Delta y \Delta z} - \frac{h^{n}_{i,j,k}}{\Delta t} S \Delta x \Delta x - q_{i-\frac{1}{2},j,k}$$

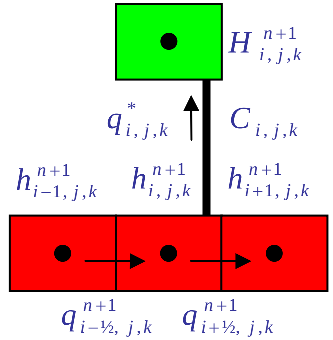
$$\Downarrow$$

$$A h^{n+1}_{i+1,j,k} + C h^{n+1}_{i,j,k} = D - q_{i-\frac{1}{2},j,k}$$

All known terms

Boundary conditions





Boundary conditions

Semipermeable or Cauchy's condition

(Leakage between aquifers, surface water – groundwater flow, etc.)

$$q^{*}_{i,j,k} = \left(h^{n+1}_{i,j,k} - H^{n+1}_{i,j,k}\right) C_{i,j,k}$$

Surface water elevation ($H^{n+1}_{i,j,k}$)

Leakage coefficient ($C_{i,j,k}$)

1D FD equations

$$-\frac{h_i^{n+1} - h_{i-1}^{n+1}}{\Delta x} Kx_{i-\frac{1}{2}} \Delta y \Delta z + \frac{h_{i+1}^{n+1} - h_i^{n+1}}{\Delta x} Kx_{i+\frac{1}{2}} \Delta y \Delta z - R - C\left(h_i^{n+1} - H_i^{n+1}\right)\Delta x \Delta y = \frac{h_i^{n+1} - h_i^{n}}{\Delta t} S_S \Delta x \Delta y \Delta z$$

$$\Downarrow$$

$$-\left(h_i^{n+1} - h_{i-1}^{n+1}\right) Kx_{i-\frac{1}{2}} + \left(h_{i+1}^{n+1} - h_i^{n+1}\right) Kx_{i+\frac{1}{2}} - \frac{\Delta x R}{\Delta y \Delta z} - C\left(h_i^{n+1} - H_i^{n+1}\right)\frac{\Delta x \Delta x}{\Delta z} = \frac{h_i^{n+1} - h_i^{n}}{\Delta t} S_S \Delta x \Delta x$$

$$\Downarrow$$

$$\left(Kx_{i+\frac{1}{2}}\right) h_{i+1}^{n+1} + \left(Kx_{i-\frac{1}{2}}\right) h_{i-1}^{n+1} + \left(-Kx_{i-\frac{1}{2}} - Kx_{i+\frac{1}{2}} + C\frac{\Delta x \Delta x}{\Delta z} - \frac{S_S \Delta x \Delta x}{\Delta t}\right) h_i^{n+1}$$

$$= R\frac{\Delta x}{\Delta y \Delta z} - CH_i^{n+1}\frac{\Delta x \Delta x}{\Delta z} - \frac{h_i^{n}}{\Delta t} S \Delta x \Delta x$$

All known terms

$$\Downarrow$$

$$Ah^{n+1}_{i+1,j,k} + Bh^{n+1}_{i-1,j,k} + Ch^{n+1}_{i,j,k} = D$$